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Research Article

Why is space-time locally Minkowskian rather than locally Euclidean?

Buket YALIN[1], Metin ARIK[1*]

[1]*Boğaziçi University Department of Physics, Faculty of Art and Sciences, İstanbul, Turkey*

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ABSTRACT

We show that the FLRW metric can be analytically continued beyond Big Bang where it describes a locally Euclidean curved space-time. Thus, we can identify the Big Bang with the creation of one time dimension, transforming a curved four dimensional space into a 3+1 dimensional space-time.

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1. INTRODUCTION

General relativity generalizes special relativity. Therefore, we may consider it as an extension of Minkowski space-time. Minkowski space-time is maximally symmetric and is the space-time of special relativity. In contrast, the de Sitter space-time[1] is also maximally symmetric, however it is a space-time of general relativity. Minkowski space-time is flat which means that it has no curvature. On the other hand, de Sitter space-time is a maximally symmetric space-time that has constant positive curvature, and it is a solution to the vacuum Einstein equations with a positive cosmological constant. de Sitter space-time is an example of closed FLRW[2]–[5] universe. It is also consistent with cosmological inflation[6] which is the idea that space underwent a fast exponential expansion in the early times of the universe. According to the standard model of cosmology, which is called the LambdaCDM[7][8] model, it is also expanding at a tiny rate today, a phenomenon which has been given the name of dark energy. Apparently, according to the accepted cosmological model[7][8] today, the space-time is not exactly Minkowski; rather, it is approaching a de Sitter space-time.

We can claim that the universe began as a locally Euclidean space-time and then has changed into a locally Minkowskian space-time since Minkowski space has SO(3) symmetry just like the 4-dimensional Euclidean space has. In this paper, we will discuss this claim based on the de Sitter solution which describes a closed FLRW universe with constant curvature.

Any four-dimensional metric can be written in isosynchronous coordinates[3] with $g_{00} = 1$ and $g_{0i} = 1$. Well known examples are the FLRW metric where x^0 is the cosmological time and the LeMaître form of the Schwarzschild metric.

***Corresponding author.**
*E-mail address: metin.arik@boun.edu.tr

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We argue that the physical Minkowskian or Euclidean time variable should be chosen as a path in the complex z plane such that the resulting four-dimensional Riemannian metric is real. Several interesting examples will be discussed.

2. EUCLIDEAN AND MINKOWSKI METRICS IN 2-DIMENSIONAL SPACE-TIME

For simplicity, we start with considerations of a 2-dimensional space-time, rather than a 4-dimensional one. The 2-dimensional transformations are basic, and they can be generalized to all dimensions. In general, the Riemannian metric is defined by

$$dS^2 = g_{mn} dX^m dX^n \tag{2.1}$$

where dX^m and dX^n are commutative and g_{mn} is a symmetric tensor which satisfies $g_{mn} = g_{nm}$. Therefore, in 2-dimensions the Riemannian metric becomes

$$dS^2 = g_{mn} dX^m dX^n = g_{11}(dX^1)^2 + 2g_{12}(dX^1 dX^2) + g_{22}(dX^2)^2 \tag{2.2}$$

where $m,n = 1,2$. The components of the metric tensor g_{mn} are functions of X^1 and X^2.

It is important that both Euclidean and Minkowski metrics have a high degree of symmetry. The Euclidean metric is

$$dS^2 = (dx^1)^2 + (dx^2)^2, \tag{2.3}$$

and the Minkowski metric is

$$dS^2 = -dt^2 + dx^2. \tag{2.4}$$

2.1. Euclidean symmetry

The Euclidean group is the group of isometries of a Euclidean space. The isometries which are distance-preserving transformations in space consist of translations, rotations, reflections, and the combinations of these. In this paper, we only concentrated on translational and rotational symmetry.

2.1.1. Translations

The translation is to move every point on a geometric figure or a space by the same distance in a given direction.

$$x^i \to x^i + a^i \tag{2.5}$$

where it has 2 parameters a^i such that $i = 1,2$.

2.1.2. Rotational symmetry

If we take $x^1 = x$ and $x^2 = y$, then the rotation matrix in 2-dimensions is

$$\begin{pmatrix} x' \\ x' \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix}. \tag{2.6}$$

This transformation has the property that the distance r from the origin is invariant, i.e. $r = r'$.

2.2. Poincaré Symmetry of Minkowski Space

Poincaré Group is the group of Minkowski space-time isometries.

2.2.1. Translations

The transformations for translations in both Euclidean and Minkowski spaces are the same. Yet in Minkowski space, there are one space dimension and one time dimension rather than two space dimensions.

$$t' = t + t_0 \tag{2.7}$$

$$x' = x + x_0 \tag{2.8}$$

2.2.2. Lorentz Transformations

Lorentz transformations in a 2-dimensional space-time can be visualized as an imaginary rotational transformation in a 2-dimensional Euclidean space. By choosing physical dimensions where the speed of light c = 1 we have the transformation matrix

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} cosh\alpha & -sinh\alpha \\ -sinh\alpha & cosh\alpha \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{2.9}$$

where $cosh\alpha = \frac{1}{\sqrt{1-v^2}}$ and $sinh\alpha = \frac{v}{\sqrt{1-v^2}}$ (v is the velocity in the units of c).

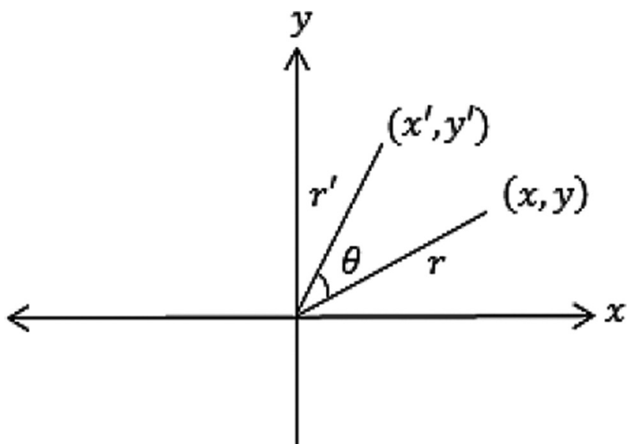

Figure 1: Rotational Symmetry in Euclidean Space.

The matrix equation above gives

$$t' = \frac{t - vx}{\sqrt{1-v^2}}, \quad x' = \frac{x - vt}{\sqrt{1-v^2}}. \tag{2.10}$$

In order to put c into the equation, one should use new variables such that

$$t \to ct, \quad x \to x, \quad v \to \frac{v}{c}. \tag{2.11}$$

In the old notation x and t both had space dimensions and v was dimensionless. Now, t has time dimension, x has space dimension and v has velocity dimension.

2.3. Special Orthogonal Group

The special orthogonal group is denoted by $SO(n)$, and is a subgroup of the orthogonal group in dimension n. $SO(n)$ is a rotational transformation matrix which has a unit determinant. As mentioned above, Euclidean space has two space dimensions while Minkowski space-time has one space dimension and one time dimension in a 2-dimensional space-time. Therefore, rotation transformations for Euclidean space are denoted by $SO(2)$, and Lorentz transformations for Minkowski space are denoted by $SO(1,1)$ in 2-dimensions.

We aim to deform the Euclidean metric by breaking the translation invariance but keeping the $SO(2)$ invariance. We want to change $SO(2)$ symmetry to $SO(1,1)$ or something like $SO(1,1)$. In order to do this, we need to define a new metric that can transform the Euclidean metric to the locally Minkowski metric at the Big Bang.

When the metric

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{2.12}$$

which is 2 × 2 symmetric matrix is diagonalized, its eigenvalues can be zero, positive or negative. Let us start from the metric

$$g_{ij} = \delta_{ij}, \tag{2.13}$$

which has the length element ds given by

$$ds^2 = \delta_{ij} dx^i dx^j = (dx^1)^2 + (dx^2)^2. \tag{2.14}$$

δ_{ij} is a tensor under $SO(2)$ symmetry.

Another tensor that transforms under $SO(2)$ symmetry is given by $x_i x_j$. It transforms such that

$$\begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}. \tag{2.15}$$

In tensor notation, the transformation is

$$\tilde{x}^i = M^i_j x^j. \tag{2.16}$$

$x_i x_j$ transforms like a tensor as it can be seen in the equation **(2.17)**:

$$\tilde{x}^i \tilde{x}^j = M^i_k x^k M^i_l x^l = M^i_k M^i_l x^i x^j. \tag{2.17}$$

Instead of the Euclidean metric **(2.13)**, we can define a new metric such that

$$g_{ij} = \delta_{ij} - f(r) x_i x_j \tag{2.18}$$

where r is rotationally invariant and is equal to

$$r = \sqrt{(x^1)^2 + (x^2)^2}. \tag{2.19}$$

We search for values of $f(r)$ such that the metric **(2.18)** changes its signature. An eigenvalue of the metric tensor smaller than zero indicates time dimension, and an eigenvalue bigger than zero indicates space dimension. Therefore, to obtain a metric which is locally Minkowskian, there must be one negative, and one positive eigenvalue of the metric tensor in a 2-dimensional space-time.

The eigenvalues of the newly defined metric **(2.18)** are

$$\lambda_1 = 1, \tag{2.20}$$

$$\lambda_2 = 1 - r^2 f(r). \tag{2.21}$$

The eigenvalue λ_1 is greater than zero, so it indicates one space dimension. It is needed to have also time dimension for a Minkowski space-time. Therefore, λ_2 must be less than zero. Nevertheless, the equation **(2.21)** can be also greater than zero, then it gives a solution for a Euclidean space. It is concluded that the metric **(2.18)** is locally Euclidean for $f(r)[(x^1)^2 + (x^2)^2] < 1$, and locally Minkowskian for $f(r)[(x^1)^2 + (x^2)^2] > 1$. This example shows that spaces which are locally Minkowskian and locally Euclidean can be described as analytical continuations of each other.

3. THE UNIVERSE BEFORE BIG BANG

In an isotropic and homogeneous universe, the universe may transform from a locally Euclidean space to the locally Minkowskian space-time at the Big Bang. In this section, we will discuss the 4-dimensional solutions of $f(r)$ that we defined in equation **(2.18)** and will identify the Big Bang as a boundary separating the locally Euclidean and locally Minkowskian regions.

Consider the metric of $\mathbf{R}^4$,

$$ds^2 = \delta_{ij} dx^i dx^j \quad (3.1)$$

where $i,j = 1,2,3,4$. The metric **(3.1)** is invariant under the 4-dimensional Euclidean group consisting of translations and rotations. Again, we define a new metric by breaking the translation invariance but keeping $SO(4)$ invariance of the metric **(3.1)** and we get

$$g_{ij} = \delta_{ij} - \frac{f(x)x_i x_j}{r^2}, \quad (3.2)$$

where $f(r)$ is dimensionless and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$.

As it is shown for the 2-dimensional case: for $f < 1$, the signature of the metric is $(+ + + +)$; and for $f > 1$, the signature of the metric is $(- + + +)$. For $f = 1$, g_{ij} is singular i.e. $det g_{ij} = 0$.

We want to identify the metric **(3.2)** with space-time and to identify its boundary with the Big Bang.

The line element is

$$ds^2 = g_{ij} dx^i dx^j = \delta_{ij} x^i x^j - \frac{f(r)x_i dx^i x_j dx^j}{r^2}. \quad (3.3)$$

Writing $\delta_{ij} x^i x^j$ in spherical coordinates, **(3.3)** becomes

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\Omega_3^2 - f(r)dr^2 \\ &= -[f(r) - 1]dr^2 + r^2 d\Omega_3^2 . \end{aligned} \quad (3.4)$$

Let us consider $f > 1$ region in the metric **(3.4)** which defines space-time. This is just the spatially closed metric given by

$$ds^2 = -dt^2 + a(t)^2 d\Omega_3^2 , \quad (3.5)$$

where t is called the cosmological time and $a(t)$ is called the scale-size of the universe. For a closed universe, space-like sections are 3-spheres and $a(t)$ is the radius of the sphere. We have

$$a(t) = r , \quad (3.6)$$

$$dt = dr \sqrt{f(r) - 1}. \quad (3.7)$$

Equation **(3.7)** also can be written as

$$dt = da \sqrt{f(a) - 1}. \quad (3.8)$$

4. FLRW MODELS

In this section, we will discuss the FLRW models. The FLRW metric is

$$ds^2 = -dt^2 + a(t)^2 \frac{d\vec{\xi}^2}{\left(1 + \frac{\kappa}{4}\vec{\xi}^2\right)^2} \quad (4.1)$$

where k can have the values of 0, ±1. So, there are three kinds of FLRW models. When $k = 1$, space-like sections are closed (S^3). When $k = 0$, space-like sections are flat (R^3). And when $k = -1$, space-like sections are open (H^3).

If the universe started from a point, $k = 0$ and $k = -1$ models are not viable. Since the Big Bang is assumed to be the boundary ($f = 1$) in the newly defined metric **(3.2)**, only the $k = 1$ model should be considered.

Let us look at the radiation dominated closed FLRW model which is given by

$$a(t) = \sqrt{t(T - t)}, \quad (4.2)$$

where $t = 0$ is the Big Bang, and $t = T$ is the Big Crunch. The graph of a with respect to t which is a semicircle in $a - t$ plane can be seen in **Figure 2**.

In FLRW coordinates, the metric **(4.2)** becomes

$$ds^2 = -dt^2 + t(T - t)d\Omega_3^2 \quad (4.3)$$

where $0 < t < T$. In this equation, t is always real. However, we should underline that the metric remains real for $t < 0$ or $t > T$.

Then apparently, according to the equation **(4.3)**, space-time has four time dimensions before the Big Bang, that is when $t < 0$.

Considering **(3.6)** and **(3.7)**, we get

$$dt = da \sqrt{f(a) - 1}. \quad (4.4)$$

From the equation **(4.2)**, t and dt can be defined as

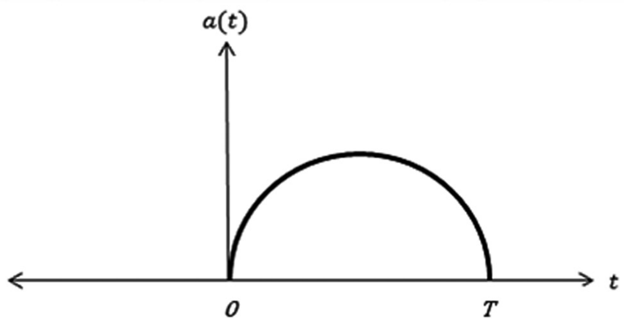

Figure 2: Graph of a vs. t.

$$t = \frac{T}{2} - \sqrt{a^2 + \frac{T^2}{4}}, \tag{4.5}$$

$$dt = -\frac{a}{\sqrt{a^2 + \frac{T^2}{4}}} da. \tag{4.6}$$

Combining equations **(4.4)** and **(4.6)**, we get

$$f = 1 + \frac{a^2}{a^2 + \frac{T^2}{4}}. \tag{4.7}$$

where **(4.7)** is always locally Minkowskian because a and T is always greater than zero which means f is always greater than 1. Thus, for this metric, choosing t as the independent variable allows us to make a real extrapolation into the locally Euclidean region. Whereas in **(4.7)** extrapolation into the locally Euclidean region happens when only imaginary values of t are considered.

5. THE RELATION BETWEEN COSMOLOGICAL MATTER-ENERGY AND EXPANSION OF THE UNIVERSE

We will review the basic concepts and concentrate on the cosmological matter-energy relation. We will calculate the relationship between a, the scale-size of the universe, and $f(r(a))$ using the continuity equation for various types of cosmological matter-energy relation.

Pressure of the universe, p, and density of the universe, ρ, are related by an equation of state which is related to the kind of matter-energy in the universe. Einstein's field equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{5.1}$$

relates the metric **(4.2)** to energy-momentum tensor (p, ρ). If there is a proportional part in $T_{\mu\nu}$ to $g_{\mu\nu}$ such that

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + F(x) g_{\mu\nu}, \tag{5.2}$$

then the conservation identity holds for $T_{\mu\nu}$.

$$\nabla_\mu T^\mu_\nu = 0 \tag{5.3}$$

implies that $F(x)$ is constant which means Λ may be included in T^μ_ν. So, then Einstein's field equations become

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{5.4}$$

For an isotropic homogeneous universe, T^μ_ν has the form

$$T^\mu_\nu = \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \tag{5.5}$$

where ρ is energy density of the universe and p is the pressure of the universe.

These lead to the following equations:

$$d\rho + 3(\rho + \mathrm{p}) \frac{da}{a} = 0, \tag{5.6}$$

$$\rho = C\left(\frac{\dot{a}^2}{a^2} + \frac{1}{a^2}\right). \tag{5.7}$$

Equation **(5.6)** is the continuity equation which is also consistent with the first law of thermodynamics. Cosmological expansions governed by Einstein's equations are reversible because Einstein's equations are time reversal invariant i.e. equations don't change under $t \rightarrow -t$.

Equation **(5.7)** is the Friedman equation where $\dot{a}$ indicates $\frac{da}{dt}$ and it holds for a closed universe. Recall that if there is Big Bang, it can be said that open universe is not physical.

Various types of cosmological matter-energy are plausible to dominate the expansion of the universe. Two of them are radiation and dark energy, or inflation dominated, universes which are indicated by

$$\mathrm{p} = \frac{1}{3}\rho \tag{5.8}$$

and,

$$\mathrm{p} = -\rho \tag{5.9}$$

respectively. Using equations **(5.8)**, **(5.9)**, and **(3.8)**; $\rho(a)$, $a(t)$, and $f(a)$ can be calculated. See **Table 1** where C' and C'' are any constants. a_0 is the minimum of scale-size of the universe when $t = 0$, and t_0 is minimum of time when $a = 0$.

Let us look at the solutions of dark energy in suitable units and put $a(t) = \cosh t$ into equation **(3.5)**. Then we get

$$ds^2 = -dt^2 + \cosh^2 t \, d\Omega_3^2. \tag{5.10}$$

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Table 1: $\rho(a)$, $a(t)$, and $f(a)$ for radiation and dark energy dominated universes

Type	Equation of State	ρ(a)	a(t)	f(a)
Radiation	$P = \frac{1}{3}\rho$	$\rho = \frac{C'}{a^4}$	$a = \sqrt{t(2C' - t)}$	$f = \frac{C'^2}{C'^2 - a^2}$
Dark energy	$p = -\rho$	$\rho = C''$	$a = t_0 cosh\left(\frac{t}{t_0}\right)$	$f = \frac{a^2}{a^2 - a_0^2}$

Table 2: Graphical representation for the relation between $a(t)$ of dark energy dominated universe and the sign of metric

***a*:**	$-\infty$	-1	1	∞
Sign of the metric:	− + + +	+ + + +	− + + +	

Equation **(5.10)** has the same space-time for $t < 0$ and $t > 0$, since it has $t \to -t$ symmetry. However, the metric **(3.5)** in terms of a for dark energy is

$$ds^2 = -\frac{1}{a^2 - 1}da^2 + a^2 d\Omega_3^2. \tag{5.11}$$

This implies that the sign of the metric **(5.11)** is $(- + + +)$ when $a < -1$ or $a > 1$which means it is locally Minkowskian and $(+ + + +)$ when $-1 < a < 1$which means locally Euclidean. This is indicated in **Table 2** for graphical representation. Also notice that the sign of the metric is symmetric under the change of $a \to -a$.

Remember $a(t) = \cosh t$ The value of $\cosh t$ is between 0 and 1 such that;

$$0 < \cosh t < 1, \tag{5.12}$$

when

$$0 < a < 1. \tag{5.13}$$

Let us think of $\cosh t$ in the complex plane by using the identity which holds for any complex variable r.

$$\cosh t = \cos it \tag{5.14}$$

Then,

$$0 < \cos it < 1 \tag{5.15}$$

where $\frac{\pi}{2} < it < 0$ and t is pure imaginary.

When we consider FLRW metric **(3.5)** with a and t as complex variables, $a(t)$ is an analytic map from a-plane to the t-plane. We want the metric **(3.5)** physically meaningful when ds^2 is real. Then we concluded that for both radiation dominated and dark energy dominated universes; locally Minkowskian universe is preceded by a locally Euclidean universe at the Big Bang. Specifically, we can define a variable τ such that $\tau = t$ for $t > 0$, and $\tau = it$ for imaginary t.

So, then the metric is

$$ds^2 = -d\tau^2 + \cosh^2 \tau \, d\Omega_3^2 \tag{5.16}$$

for $\tau > 0$, and

$$ds^2 = -d\tau^2 + \cos^2 \tau \, d\Omega_3^2 \tag{5.17}$$

for $\tau < 0$. This metric is continuous at $\tau = 0$.

6. CONCLUSION

We conclude that for both the radiation dominated and the dark energy dominated universes, the locally Minkowskian universe is preceded by a locally Euclidean universe at the Big Bang. Here we define the Big Bang where the scale-size for the Minkowskian interpretation is minimum. To achieve this, we have extended the variable t of cosmological time and/or the variable a of scale-size to complex values by using analytic continuation.

According to the celebrated Whitney theorem, any curved Riemannian space can be embedded into $\mathbf{R}^8$. We need to consider 4-dimensional complex space-time as $\mathbf{C}^4$ which is a 4-dimensional flat complex space that is isomorphic to $\mathbf{R}^8$. Then Whitney theorem suggests that General Relativity may be formulated in a 4-dimensional flat complex space. Yet we, humans, can measure only its real part. This, of course, is a subject of future investigation.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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